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AN EXPLICIT DESCRIPTION OF THE RELATIVE $\mathrm{SL}(4,\mathbb{C})$-CHARACTER VARIETY OF THE PROJECTIVE LINE

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Introduction

- n: a positive integer.
- $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$: a k-tuple of partitions of n.
- $(\mathcal{C}_1, \dots, \mathcal{C}_k)$: a k-tuple of semisimple conjugacy classes such that $\mathcal{C}_i \subset \mathrm{SL}(n,\mathbb{C})$ and the multiplicities of eigenvalues of matrices in $\mathcal{C}_i$ are given by $\mu^i = (\mu^i_1, \mu^i_2, \dots)$.

Character variety of the projective line

$$\mathcal{R}^{\boldsymbol{\mu}}_{n,k} := \{(M_1, \dots, M_k) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \mid M_1 \cdots M_k = \mathrm{Id}\} /\!/ \mathrm{SL}(n,\mathbb{C}).$$

Suppose that $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is generic. Then, we have

1. $\mathcal{R}^{\boldsymbol{\mu}}_{n,k}$ is non-singular;
2. $\mathcal{R}^{\boldsymbol{\mu}}_{n,k}$ has a holomorphic symplectic structure.

The list of the cases where $\dim \mathcal{R}^{\boldsymbol{\mu}}_{n,k} = 2$

$\boldsymbol{\mu} = ((11),(11),(11),(11))$	[1]
$\boldsymbol{\mu} = ((111),(111),(111))$	[3]
$\boldsymbol{\mu} = ((1111),(1111),(22))$	main result
$\boldsymbol{\mu} = ((111111),(222),(33))$	unknown.

Problem

Give explicit descriptions of these character varieties. Moreover, give compactifications of these character varieties.

A conjecture due to Simpson (Motivation)

There exists a non-singular compactification of $\mathcal{R}^{\boldsymbol{\mu}}_{n,k}$ such that the boundary complex is a simplicial decomposition of sphere $S^{\dim \mathcal{R}^{\boldsymbol{\mu}}_{n,k}-1}$.

$((11),(11),(11),(11))$ and $((111),(111),(111))$

Invariants of the $\mathrm{SL}(2,\mathbb{C})$-action.

$x_i := \mathrm{Tr}(M_k M_j)$ $(i,j,k) =$ a cyclic permutation of $(1,2,3)$,
$a_i := \mathrm{Tr} M_i$ $(i = 1,2,3)$, $a_4 := \mathrm{Tr}(M_3 M_2 M_1)$.

Relation.

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\boldsymbol{a})x_1 - \theta_2(\boldsymbol{a})x_2 - \theta_3(\boldsymbol{a})x_3 + \theta_4(\boldsymbol{a})$$

where

$$\theta_i(\boldsymbol{a}) = a_i a_4 + a_j a_k \quad (i,j,k) = \text{ a cyclic permutation of } (1,2,3),$$
$$\theta_4(\boldsymbol{a}) = a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4.$$

Compactification.

Put $x_1 := x/w, x_2 := y/w, x_3 := z/w$. Then, we obtain the following homogeneous polynomial

$$xyz + x^2w + y^2w + z^2w - \theta_1(a)xw^2 - \theta_2(a)yw^2 - \theta_3(a)zw^2 + \theta_4(a)w^3 = 0.$$

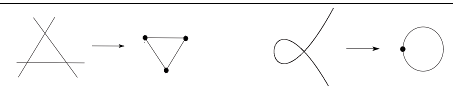
Fricke–Klein, et al

The compactification $\overline{\mathcal{R}^{\boldsymbol{\mu}}_{2,4}}$ is a del Pezzo surface of degree 3. The divisor at infinity is a triangle.

In a similar way as in the case $((11),(11),(11),(11))$, we have

Lawton

The compactification $\overline{\mathcal{R}^{\boldsymbol{\mu}}_{3,3}}$ is a del Pezzo surface of degree 3. The divisor at infinity is a nodal rational curve.



$((1111),(1111),(22))$

Normalization of matrices.

$$\left(\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}, M_3 \right)$$

where $M_3^{-1} = M_1 M_2$.

- $\boldsymbol{b} = (b_1, \dots, b_4)$: the eigenvalues of matrices of $\mathcal{C}_2$.
- c_1, c_2: the eigenvalues of matrices of $\mathcal{C}_3$.

Relations.

$$\begin{aligned} &\det(\lambda \mathrm{Id} - M_2) = \lambda^4 + f_3(\boldsymbol{b})\lambda^3 + f_2(\boldsymbol{b})\lambda^2 + f_1(\boldsymbol{b})\lambda^1 + 1, \\ &(M_3 - c_1^{-1})(M_3 - c_2^{-1}) = \mathbf{0}, \\ &\mathrm{Tr}(M_3) = -2c_1^{-1} - 2c_2^{-1}. \end{aligned} \tag{1}$$

Invariants of the torus action.

$s_{ii} := x_{ii}$,
$s_{ij} := x_{ij}x_{ji}$ $(i < j)$,
$s_{ijk} := x_{ij}x_{jk}x_{ki}$ $(i < j, i < k, j \neq k)$,
$s_{ijkl} := x_{ij}x_{jk}x_{kl}x_{li}$ $(i < j, i < k, i < l, j \neq k, k \neq l, l \neq j)$.

By the equations (1), we have the equations of s_{ii}, s_{ij}, s_{ijk}, and s_{ijkl}. We may solve these equations for the variables by

$$s_{231}, s_{31}, \text{ and } s_{11}.$$

By the following equations

$$s_{ijk}s_{ikj} = s_{ij}s_{jk}s_{ik},$$

we have an equation of s_{231}, s_{31}, and s_{11}. The equation defines the character variety $\mathcal{R}_1$ in $\mathbb{C}^3$.

Compactification.

Put $s_{231} := w/x^3$, $s_{31} := z/x^2$, $s_{11} := y/x$. Then, we have the following polynomial

$$f^{\boldsymbol{\mu}}_{1,3} := Aw^2 + \left(zf_1^{(1)}(x,y) + f_3^{(1)}(x,y)\right)w + \left(Bz^3 + z^2f_2^{(0)}(x,y) + zf_4^{(0)}(x,y) + f_6^{(0)}(x,y)\right)$$

where $f_i^{(j)}$ is a homogeneous polynomial of degree i. We denote by $\overline{\mathcal{R}^{\boldsymbol{\mu}}_{1,3}}$ the locus defined by $f^{\boldsymbol{\mu}}_{1,3}$ in $\mathbb{P}(1,1,2,3)$.

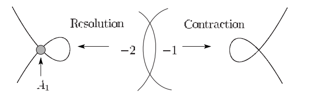
$$\phi\colon \overline{\mathcal{R}^{\boldsymbol{\mu}}_{1,3}} \xrightarrow{2:1} \mathbb{P}(1,1,2)$$
$$[x:y:z:w] \longmapsto [x:y:z].$$

Singularity (of type A_1).

$$[x:y:z] = \left[0 : 1 : \frac{(a_4 - a_1)(a_4 - a_2)}{(a_2 - a_3)(a_3 - a_4)}\right] \in \mathbb{P}(1,1,2).$$

The main result

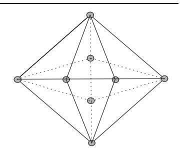
The compactification $\overline{\mathcal{R}^{\boldsymbol{\mu}}_{1,3}}$ is a singular del Pezzo surface of degree 1. The surface has a singularity of type A_1. The divisor at infinity is a nodal rational curve passing through the singularity.



Higher dimensional case

Suppose that $n = 2$, $k = 5$, and $\boldsymbol{\mu} = ((11),(11),(11),(11),(11))$. Then $\dim \mathcal{R}^{\boldsymbol{\mu}}_{2,5} = 4$.
We obtain the boundary complex which is a simplicial decomposition of sphere S^3 [2].

The subcomplex associated to S^2 in S^3:



References

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